Project 3 - TMA4320

Numerical physics project
Compact stars

Sondre Duna Lundemo[a]

[a] *Department of physics,*
Norwegian University of Science and Technology,
N-7491 Trondheim, Norway.

Abstract: The Lane-Emden equation and the TOV-equations are solved analytically and numerically for a variety of cases. Theoretical aspects of the equations, the underlying physics and the numerical methods is discussed. The code is available in an accompanying `python` file.

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1 Introduction

A star is a cloud of partially ionized gas. Exciting and extreme physics take place near such objects, and to understand these processes it is natural to start by examining properties we are able to measure: its *mass* and *radius*. Of particular interest the relation between the two. This project will be centred around the *mass-radius relation* for stars, through solving the equations of stellar structure from [1]:

$$\frac{dm}{dr} = 4\pi\rho(r)r^2 \tag{1.1}$$

$$\frac{dP}{dr} = -\frac{G\epsilon(r)m(r)}{c^2r^2}\left[1+\frac{P}{\epsilon(r)}\right]\left[1+\frac{4\pi r^3P(r)}{m(r)x^2}\right]\left[1-\frac{2Gm(r)}{c^2r}\right]^{-1}, \tag{1.2}$$

where c is the speed of light, $\rho(r)$ is the mass density of the star, $\epsilon(r)$ is the energy density, $m(r)$ is the mass, $P(r)$ is the pressure, and G is Newtons gravitational constant. These equations are known as the Tolman-Oppenheimer-Volkov (TOV) equations. The Newtonian versions of these equations are much simpler and read

$$\frac{dm}{dr} = 4\pi\rho(r)r^2 \tag{1.3}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\epsilon}\frac{dP}{dr}\right) = -\frac{4\pi G\epsilon(r)}{c^4} \tag{1.4}$$

2 Theory

2.1 Problem 1 - The Lane-Emden equation

a) Derivation of the Lane-Emden equation

It is useful to rewrite equation (1.4) to a dimensionless form. To do this, we introduce an equation of state for the particles of the star of the form

$$P = K\epsilon^\gamma,$$

where $\gamma = 1+\frac{1}{n}$, and n is the *polytropic index*. Define the dimensionless parameters θ and ξ through

$$\epsilon = \epsilon_0\theta^n \ \ , r = a\xi$$

where ϵ_0 is the energy density at $r = 0$ and

$$a = \sqrt{\frac{(n+1)Kc^4\epsilon^{1/n-1}}{4\pi G}}. \tag{2.1}$$

We start by considering the derivative of the pressure with respect to r

$$\frac{dP}{dr} = \frac{dP}{d\xi}\frac{d\xi}{dr} = \frac{dP}{d\epsilon}\frac{d\epsilon}{d\theta}\frac{d\theta}{d\xi}\frac{d\xi}{dr} = K\gamma\epsilon^{\gamma-1}\cdot\epsilon_0 n\theta^{n-1}\cdot\frac{d\theta}{d\xi}\cdot\frac{1}{a}$$
$$= K\gamma\epsilon_0^{\gamma-1}\theta^{n(\frac{1}{n}+1-1)}\cdot\epsilon_0 n\theta^{n-1}\cdot\frac{d\theta}{d\xi}\cdot\frac{1}{a} = \frac{K\gamma n\epsilon_0^{\gamma}}{a}\theta^n\frac{d\theta}{d\xi}.$$

The left hand side of equation (1.4) then reduces to

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\epsilon}\frac{dP}{dr}\right) = \frac{1}{a^2\xi^2}\frac{d}{d\xi}\left(\frac{a^2\xi^2}{\epsilon_0\theta^n}\frac{K\gamma n\epsilon_0^{\gamma}}{a}\theta^n\frac{d\theta}{d\xi}\right)\frac{d\xi}{dr}$$
$$= \frac{\gamma n K\epsilon_0^{\gamma-1}}{a^2\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right).$$

We move all constants to the right hand side of the equation to get the following

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = -\frac{4\pi G\epsilon_0\theta^n a^2}{\gamma Knc^4\epsilon_0^{\gamma-1}}$$
$$= -\frac{4\pi G}{\gamma nKc^4\epsilon_0^{\gamma-2}}\frac{(n+1)Kc^4\epsilon_0^{1/n-1}}{4\pi G}\theta^n$$
$$= -\frac{\theta^n(n+1)}{(1/n+1)n\epsilon_0^{1/n+1-2}}\epsilon_0^{1/n-1} = -\theta^n,$$

where we in the second transition used the definition of a in equation (2.1), and in the following step the definition of γ in terms of n. This ultimately yields the *Lane-Emden equation*

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = -\theta^n.} \tag{2.2}$$

b) Solution of Lane-Emden equation for $n = 0$

With $n = 0$, equation (2.2) reduces to

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = -1,$$

which is solvable by separation and direct integration

$$\int d\xi\,\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = -\int d\xi\xi^2 \Rightarrow \xi^2\frac{d\theta}{d\xi} = -\frac{1}{3}\xi^3 + C_1$$
$$\int d\xi\,\frac{d\theta}{d\xi} = \int d\xi\,\left(-\frac{1}{3}\xi + C_1\frac{1}{\xi^2}\right) \Rightarrow \theta(\xi) = -\frac{1}{6}\xi^2 - \frac{C_1}{\xi} + C_2.$$

It is evident that we mush have $C_1 = 0$ to avoid a singular energy density at $r = 0$. When imposing the boundary condition $\theta(0) = 1$, we obtain $C_2 = 1$. This evidently also makes the equation satisfy the other boundary condition of $\theta'(0) = 0$. The solution is therefore

$$\boxed{\theta(\xi) = \frac{1}{6}\left(6 - \xi^2\right)}$$

c) Solution of Lane-Emden equation for $n = 1$

For $n = 1$, equation (2.2) reads

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = -\theta,$$

which is significantly more complicated than when $n = 0$. To simplify the equation, write $\theta = u(\xi)/\xi$. This yields

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d}{d\xi}\frac{u}{\xi} = \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{\xi u' - u}{\xi^2}\right) = \frac{1}{\xi^2}\left(\xi u'' + u' - u'\right) = -\frac{u}{\xi} \Leftrightarrow$$
$$u'' = -u,$$

which is the familiar equation of a harmonic oscillator, which has solutions

$$u(\xi) = \alpha \sin\xi + \beta\cos\xi.$$

From the definition of u we can solve for θ, namely

$$\theta(\xi) = \frac{1}{\xi}\left(\alpha\sin\xi + \beta\cos\xi\right).$$

We proceed by imposing the boundary conditions $\theta(0) = 1$ and $\theta'(0) = 0$. Once again, to obtain physically admissible solutions, we have to avoid the singularity at $r = 0$, which occurs whenever $\beta \neq 0$. With $\beta = 0$, the first boundary condition is satisfied with $\alpha = 1$ by virtue of the well known limit

$$\theta(0) = \lim_{\xi\to 0}\alpha\frac{\sin\xi}{\xi} = \alpha = 1.$$

By treating the limit of $\theta(\xi)$ as $\xi \to 0$ with *L'Hôpital's rule*, it is evident that the solution with $\alpha = 1, \beta = 0$ also satisfy the second boundary condition

$$\theta'(0) = \lim_{\xi\to 0}\frac{\xi\cos\xi - \sin\xi}{\xi^2} = \lim_{\xi\to 0}\frac{\cos\xi - \xi\sin\xi - \cos\xi}{2\xi} = 0.$$

The solution therefore reads

$$\boxed{\theta(\xi) = \frac{\sin\xi}{\xi} = \operatorname{sinc}\xi.}$$

2.2 Problem 2 - The TOV-equations

a) Derivation of dimensionless TOV-equation

To derive a dimensionless form of equation (1.2), we introduce the following dimensionless parameters.

$$\bar{P} = \frac{P}{\rho_0 c^2}, \quad , x = \frac{r}{R} \quad , \alpha = \frac{r_s}{R}$$

Here, R is the radius of the star defined by $M = \frac{4\pi R^3 \rho_0}{3}$, and $r_s = \frac{2GM}{c^2}$ is the *Schwarzschild radius* . We begin by writing the left hand side of equation (1.2) in dimensionless form

$$\frac{dP}{dr} = \frac{dP}{d\bar{P}}\frac{d\bar{P}}{dx}\frac{dx}{dr} = \rho_0 c^2 \frac{d\bar{P}}{dx}\frac{1}{R} = \frac{\rho_0 c^2}{R}\frac{d\bar{P}}{dx}.$$

We proceed by dividing both sides with the constants in front of the derivative, and then rewriting the first factor of the right hand side of equation (1.2).

$$\begin{aligned}
-\frac{R}{\rho_0 c^2}\frac{G\epsilon(r)m(r)}{c^2 r^2} &= -\frac{RG\rho_0 c^2}{\rho_0 c^2 r^2}\frac{4\pi r^3}{3}\rho_0 = -\frac{4\pi R^2}{3}Gx = -\frac{4\pi \rho_0 R^3}{3}G\frac{x}{R} \\
&= -\frac{GM}{c^2}\frac{x}{R} = -\frac{1}{2}r_s\frac{x}{R} = -\frac{1}{2}\alpha x
\end{aligned}$$

Then, we rewrite the rest of the expression, one parentheses at the time.

$$\begin{aligned}
\left[1 + \frac{P(r)}{\epsilon(r)}\right] &= \left[1 + \frac{\rho_0 c^2 \bar{P}}{\rho_0 c^2}\right] = \left[1 + \bar{P}\right] \\
\left[1 + \frac{4\pi r^3 P(r)}{m(r)c^2}\right] &= \left[1 + \frac{4\pi r^3 \rho_0 c^2 \bar{P}}{\frac{4\pi r^3}{3}\rho c^2}\right] = \left[1 + 3\bar{P}\right] \\
\left[1 - 2\frac{Gm(r)}{c^2 r}\right]^{-1} &= \left[1 - \frac{2GM}{c^2}\frac{r^3}{R^3}\frac{1}{r}\right]^{-1} = \left[1 - r_s\frac{r^2}{R^3}\right]^{-1} \\
&= \left[1 - \left(\frac{r_s}{R}\right)\left(\frac{r}{R}\right)^2\right]^{-1} = \left[1 - \alpha x^2\right]^{-1},
\end{aligned}$$

where we in the last expression used the fact that $\frac{m(r)}{M} = \frac{r^3}{R^3}$ because of spherical symmetry. This leaves us with the dimensionless version of equation (1.2):

$$\frac{d\bar{P}}{dx} = -\frac{1}{2}\alpha x \left[1 + \bar{P}\right]\left[1 + 3\bar{P}\right]\left[1 - \alpha x^2\right]^{-1} \tag{2.3}$$

b) General solution of the TOV-equation

The radius of the star is the distance from the origin at which the pressure is 0. Therefore, when $x = 1$, $\bar{P} = 0$. This provides an initial condition which can be used to solve equation (2.3) analytically. Division upon $\left[1+\bar{P}\right]\left[1+3\bar{P}\right]$ and integration with respect to x yields

$$\int_0^{\bar{P}(x)} \frac{d\bar{P}}{\left(1+\bar{P}\right)\left(1+3\bar{P}\right)} = -\int_1^x d\xi\, \frac{1}{2}\alpha\xi(1-\alpha\xi^2)^{-1}$$

We solve each side of the equation in turn. By partial fraction decomposition of the left hand side we obtain

$$\begin{aligned}\int_0^{\bar{P}(x)} \frac{d\bar{P}}{\left(1+\bar{P}\right)\left(1+3\bar{P}\right)} &= \frac{3}{2}\int_0^{\bar{P}(x)} d\bar{P}\frac{1}{3\bar{P}+1} - \frac{1}{2}\int_0^{\bar{P}(x)} d\bar{P}\frac{1}{\bar{P}+1} \\ &= \frac{1}{2}\ln\left(\frac{3\bar{P}+1}{\bar{P}+1}\right).\end{aligned}$$

On the right hand side, a substitution solves the integral

$$\begin{aligned}-\frac{1}{2}\int_1^x d\xi\frac{\alpha\xi}{1-\alpha\xi^2} &= -\frac{1}{2}\int_\alpha^{\alpha x^2} \frac{d\eta}{1-\eta} = \frac{1}{4}\Big|_\alpha^{\alpha x^2} \ln\left(1-\eta\right) \\ &= \frac{1}{4}\ln\left(\frac{1-\alpha x^2}{1-\alpha}\right).\end{aligned}$$

The solution for $\bar{P}$ is obtained by reordering the equation

$$\begin{aligned}\frac{3\bar{P}+1}{\bar{P}+1} &= \sqrt{\left(\frac{1-\alpha x^2}{1-\alpha}\right)} \Leftrightarrow 3\bar{P}+1 = \sqrt{\left(\frac{1-\alpha x^2}{1-\alpha}\right)}(\bar{P}+1) \\ \bar{P}\left(3-\sqrt{\left(\frac{1-\alpha x^2}{1-\alpha}\right)}\right) &= \sqrt{\left(\frac{1-\alpha x^2}{1-\alpha}\right)} - 1 \Leftrightarrow \\ \bar{P}\left(\frac{3\sqrt{1-\alpha}-\sqrt{1-\alpha x^2}}{\sqrt{1-\alpha}}\right) &= \frac{\sqrt{1-\alpha x^2}-\sqrt{1-\alpha}}{\sqrt{1-\alpha}}\end{aligned}$$

Dividing both sides with the factor in front of $\bar{P}$ ultimately yields

$$\bar{P}(x) = \frac{\sqrt{1-\alpha x^2}-\sqrt{1-\alpha}}{3\sqrt{1-\alpha}-\sqrt{1-\alpha x^2}}. \tag{2.4}$$

c) Approximate solution of the TOV-equation

For negligible relativistic effects, equation (2.3) reduces to

$$\frac{d\bar{P}}{dx} = -\frac{1}{2}\alpha x$$

with solution

$$\bar{P}(x) = \frac{1}{4}\alpha(1-x^2).$$

This can also be obtained by expanding equation (2.4) in the parameter α, which is a *small* parameter when relativistic effects are small. By using the fact that

$$\sqrt{1-x} = 1 - \frac{1}{2}x + \mathcal{O}(x^2),$$

we obtain

$$\bar{P}(x) = \frac{1 - \frac{1}{2}\alpha x^2 - 1 + \frac{1}{2}\alpha + \mathcal{O}(\alpha^2)}{3 - \frac{3}{2}\alpha - 1 + \frac{1}{2}\alpha x^2 + \mathcal{O}(\alpha^2)}.$$

The fact that $x \leq 1$ (since $r \leq R$) justifies treating αx^2 as a small quantity when $\alpha \ll 1$. By neglecting terms of $\mathcal{O}(\alpha^2)$ in the numerator, and $\mathcal{O}(\alpha)$ in the denominator, the expression simplifies to

$$\bar{P}(x) \approx \frac{\frac{1}{2}\alpha - \frac{1}{2}x^2}{2} = \frac{1}{4}\alpha\left(1 - x^2\right),$$

which is exactly the same as the solution from the TOV-equation when relativistic effects are neglected all together from the beginning.

3 Numerical Methods

3.1 Problem 3 - Theoretical aspects of Euler's method

a) Eulers method - truncation error

We want to approximate the solution of the initial value problem

$$y'(x) = f(x, y), \;\; y(x_0) = y(0).$$

Suppose we have partitioned our axis in points $\{x_i\}_{i=0}^{N}$, and calculated an approximation $y_i \approx y(x_i)$. Then the next step is given by

$$y_{i+1} = y_i + hf(x_i, y_i), \tag{3.1}$$

where $x_i = x_0 + hi$. Define the local truncation error of the first step by

$$\tau_1 = y(x_1) - y_1.$$

Claim *The local truncation error of Euler's method is $\mathcal{O}(h^2)$.*
Proof We show this only for the first step, but the computation is completely analogous for an arbitrary step. If we assume that the exact solution of the IVP is a *twice continuously differentiable* function, we can expand $y(x_1) = y(x_0 + h)$ with respect to x_0 with a second order error term by virtue of Taylor's theorem

$$y(x_0 + h) = y(x_0) + hy(x_0) + \frac{1}{2}h^2 y''(\xi),$$

where $\xi \in (x_0, x_0 + h)$. Inserting into the equation for the truncation error

$$\begin{aligned}
\tau_1 = y(x_1) - y_1 &= y(x_0) + hy(x_0) + \frac{1}{2}h^2 y''(\xi) - y_1 \\
&= y(x_0) + hy(x_0) + \frac{1}{2}h^2 y''(\xi) - (y_0 + hy_0) \\
&= y(x_0) + hy(x_0) + \frac{1}{2}h^2 y''(\xi) - (y(x_0) + hy(x_0)) \\
&= \frac{1}{2}h^2 y''(\xi)
\end{aligned}$$

where we have used the fact that the first approximation of course is exact $y_0 = y(x_0)$, and we have expanded y_1 from (3.1). □

b) Euler's method - order

Claim *Euler's method has order* 1*, that is, the global error after* n *steps is* $\mathcal{O}(h)$*.*
Proof Suppose that the function $f(x,y)$ in the IVP satisfy a Lipschitz condition with respect to its second argument, that is, there exists $L \in \mathbb{R}$ such that for every $y, \eta \in \mathbb{R}$

$$|f(x,y) - f(x,\eta)| \leq L|y-\eta| \quad , x \in [x_0, x_N].$$

Then, by using the definition of the global error as $e_n = y_n - y(x_n)$, we have

$$\begin{aligned} |e_{n+1}| = |y_{n+1} - y(x_{n+1})| &= |y_n + hf(x_n, y_n) - y(x_n) - hf(x_n, y(x_n)) - \tau_{n+1}| \\ &\leq |y_{n+1} - y(x_n)| + h|f(x_n, y_n) - f(x_n, y(x_n))| + |\tau_{n+1}| \\ &\leq |y_{n+1} - y(x_n)| + hL|y_{n+1} - y(x_n)| + |\tau_{n+1}| = (1+hL)|e_n| + |\tau_{n+1}|, \end{aligned}$$

where we in the last line introduced the Lipschitz-condition on f. Suppose now that the second derivative of the exact solution $y(x)$ is bounded, that is, define

$$M = \max_{\xi \in [x_0, x_N]} |y''(\xi)|,$$

where $M < \infty$. We may then express the local truncation error τ_{n+1} as $\frac{1}{2}h^2 M$ from the previous . It now follows by induction that the global error after N steps is

$$|e_N| \leq (1+hL)|e_0| + \frac{1}{2}h^2 M \sum_{k=0}^{N-1} (1+hL)^k$$

where $|e_0| = 0$ since we assume we have the correct initial value. The sum of the series can easily be evaluated as a geometric series

$$\sum_{k=0}^{N-1} (1+hL)^k = \frac{(1+hL)^N - 1}{1 + hL - 1} = \frac{1}{hL}\left((1+hL)^N - 1\right).$$

By noting that $1 + hL \leq e^{hL}$ (the first two terms of its Taylor series), we have a bound which enables us to rewrite the expression as

$$\frac{1}{hL}\left((1+hL)^N - 1\right) \leq \frac{1}{hL}\left(e^{hLN} - 1\right) = \frac{1}{hL}\left(e^{L(x_N - x_0)} - 1\right),$$

where we have made use of the fact that $h = \frac{x_N - x_0}{N}$. By plugging all back into the expression for $|e_N|$, we get

$$|e_N| \leq \frac{1}{2}h^2 M \frac{1}{hL}\left(e^{L(x_N - x_0)} - 1\right) = \frac{hM}{2L}\left(e^{L(x_N - x_0)} - 1\right),$$

which is $\mathcal{O}(h)$ as initially claimed. □

3.2 Problem 3 - Numerical solution of the Lane-Emden equation

c) Rewriting Lane-Emden equation to linear system

Let $\chi = \theta'$. By writing out the derivatives in the Lane-Emden equation as

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = \frac{d^2\theta}{d\xi^2} + \frac{2}{\xi}\frac{d\theta}{d\xi} = -\theta^n,$$

we can express χ' as

$$\frac{d\chi}{d\xi} = \frac{d^2\theta}{d\xi^2} = -\theta^n - \frac{2}{\xi}\frac{d\theta}{d\xi} = -\theta^n - \frac{2}{\xi}\chi.$$

By stacking χ and θ in a column vector, $\boldsymbol{\omega}(\xi) = \begin{bmatrix}\theta(\xi) \; \chi(\xi)\end{bmatrix}^T$, we can rewrite the second order differential equation into a linear system of differential equations

$$\frac{d\boldsymbol{\omega}}{d\xi}(\xi) = \begin{bmatrix}\theta(\xi) \\ \chi(\xi)\end{bmatrix}' = \boldsymbol{f}(\xi, \boldsymbol{\omega}) = \begin{bmatrix}\chi(\xi) \\ -\theta^n(\xi) - \frac{2}{\xi}\chi(\xi)\end{bmatrix}. \tag{3.2}$$

Solving the system numerically with Euler's method does not change the overall structure of the solving scheme. That is, we still have

$$\boldsymbol{\omega}_{n+1} = \boldsymbol{\omega}_n + h\boldsymbol{f}(\xi_n, \boldsymbol{\omega}_n), \tag{3.3}$$

for $n = 0, \ldots, N$ and now with initial value $\boldsymbol{\omega}_0 = \begin{bmatrix}\theta(0) \; \theta'(0)\end{bmatrix}^T = \begin{bmatrix}1 \; 0\end{bmatrix}^T$. When solving the equation numerically one finds a problem for $\xi = 0$ whence χ' seems to be singular. The singularity is however removable, which is seen by noting that

$$\left.\frac{d\chi}{d\xi}\right|_{\xi=0} = -\underbrace{\theta^n(0)}_{=1} - \frac{2}{\xi}\chi(0) = -1 - 2\left.\frac{d\chi}{d\xi}\right|_{\xi=0} \Rightarrow \left.\frac{d\chi}{d\xi}\right|_{\xi=0} = -\frac{1}{3},$$

where the $\frac{0}{0}$-expression is handled using L'Hôpital's rule, as usual.

d) Solving the Lane-Emden equation with Euler's method

When choosing a step length, it is convenient to specify a tolerance, which is such that we accept the step length when the error is less than the tolerance. This idea may be incorporated in an *adaptive* method, or - as in this case - when the analytical solution is known, as an *a posteriori* check criterion. In order to get a rough estimate of how large the deviation is for a given h, we compare both the ℓ_2 and ℓ_∞ norms of the deviations, namely

$$\texttt{err}_{\ell_2} = \sqrt{\sum_{i=0}^{N}(\theta(\xi_i) - \theta_i)^2}\ , \texttt{err}_{\ell_\infty} = \max_{0\le i\le N}|\theta(\xi_i) - \theta_i|\,.$$

If we choose a `tol` $= 0.01$, we see from the listing below that a step length of size 0.01 gives an acceptable deviation measured in the ℓ_∞ norm. Further, it is seen from the plots below that this particular numerical solution very well resembles the features of the analytical solution. It should be marked however, that the notion of error will be addressed more thoroughly in the coming exercises, as this only gives a rough indication.

```
+----------+-----------+--------------+
|    h     | l_2 error | l_infty error |
+----------+-----------+--------------+
| 3.00e-01 |  1.18e-01 |    5.64e-02   |
| 1.00e-01 |  5.80e-02 |    1.56e-02   |
| 1.00e-02 |  1.71e-02 |    1.41e-03   |
+----------+-----------+--------------+
```

Listing 1. Deviations from the analytical solution for different values of h with Euler's method.

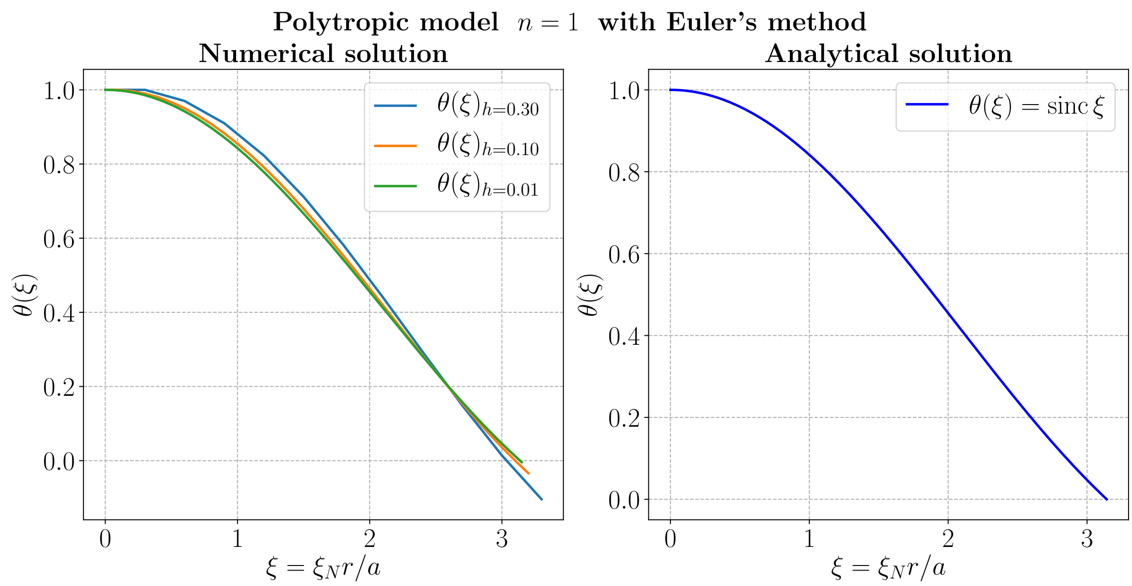

Figure 1. The figure shows the plot of the numerical and analytical solution of the Lane-Emden equation for $n = 1$, solved with Euler's method.

e) Comparing the non- and ultra-relativistic cases of the Lane-Emden equation

Since $h = 0.01$ seemed a reasonable choice in the case when $n = 1$, the same h is used when solving the Lane-Emden equation in the non-relativistic, and ultra-relativistic case. The results are shown below, together with their associated $\theta^n(\xi)$ for illustration.

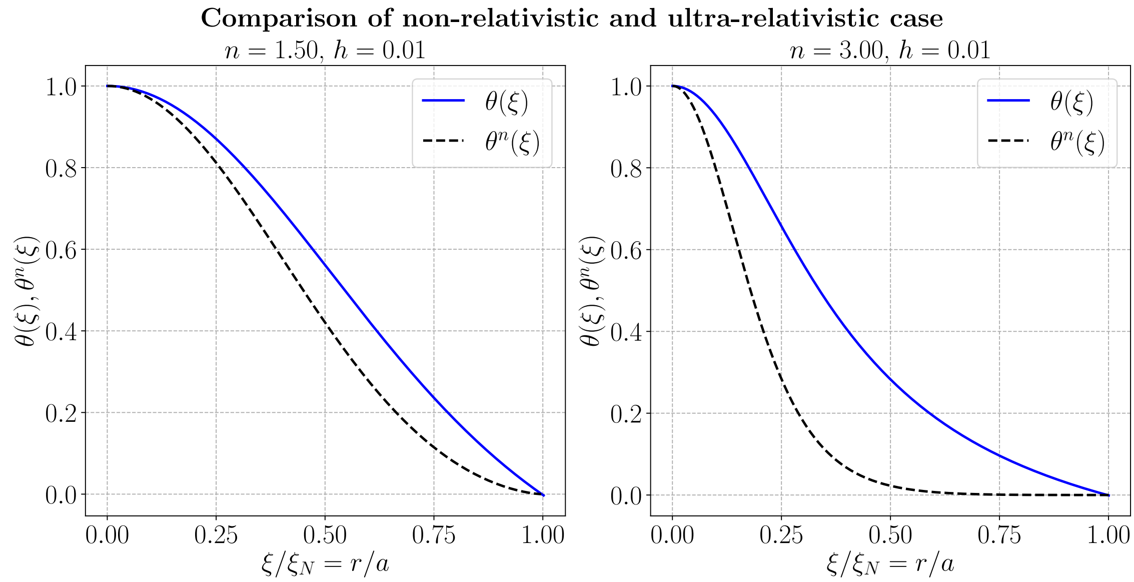

Figure 2. The figure shows the plot of the numerical solution of the Lane-Emden equation for $n = 1.5$ and $n = 3$, solved with Euler's method. The plot of $\theta^n(\xi)$ is included for comparison.

f) Solving the Lane-Emden equation with RK4 method

We resort to the same strategy for finding an optimal h as in problem 3d when solving the equation with the fourth order Runge Kutta method (RK4). The deviations are drastically reduced, since we in this case can take $h = 0.1$ and have a error in the ℓ_∞ norm less than the tolerance of 0.01. The agreement between the numerical and analytical results are also evident from the plot in figure 3.

```
+----------+-----------+---------------+
|     h    | l_2 error | l_infty error |
+----------+-----------+---------------+
| 3.00e-01 |  6.96e-05 |    4.51e-05   |
| 1.00e-01 |  1.26e-06 |    5.56e-07   |
| 1.00e-02 |  4.10e-10 |    5.56e-11   |
+----------+-----------+---------------+
```

Listing 2. Deviations from the analytical solution for different values of h with RK4-method.

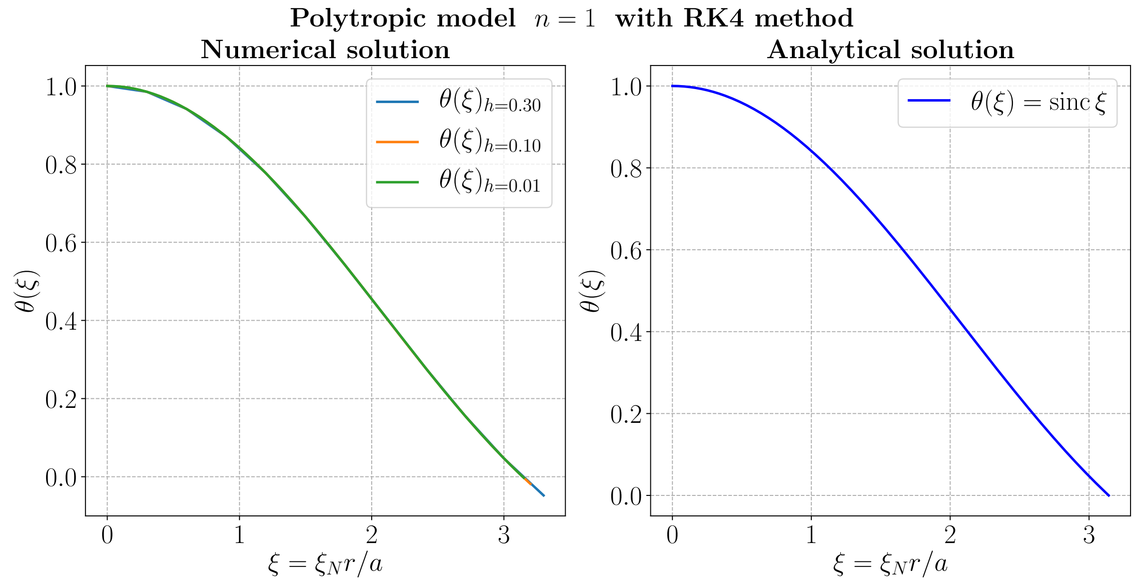

Figure 3. The figure shows the plot of the numerical and analytical solution of the Lane-Emden equation for $n = 1$, solved with RK4-method.

g) Error analysis

Since we are given the value of ξ_N which makes $\theta(\xi_N) = 0$ for both the non-relativistic and ultra-relativistic case, an estimate for the global error is thus

$$\epsilon_N = |\theta(\xi_N) - \theta_N| = |\theta_N|.$$

In order to evaluate the numerical solution at the root, when the root not necessarily is an integration point, the numerical solution is linearly interpolated at ξ_N. The linearisation is done with respect to the two last integration points. This is a reasonable approximation since the numerical solver is programmed to stop when the sign of theta changes, that is, when we reach the root. Therefore, the interpolation abscissae are the two points closest to ξ_N.

In order to easily see how the error e_N varies with h, the plots uses logarithmic scales of the axes. If we expect the error to be proportional to h to some power α, then

$$e_N = Ch^\alpha \Rightarrow \log e_N = \log C + \alpha \log h,$$

thus $\log e_N$ is a linear function of $\log h$ and we can through plotting these quantities easily find the α as the slope of a linear function. Overall, the plots below of $e_N(h)$ indeed show the expected linear behaviour. For Euler's method, by performing a linear regression on the data set $\{\log \epsilon_N, \log h\}_{h=0.0001}^{0.1}$, one obtains an estimate of alpha as

$$\alpha_{\text{Euler}} \approx 1.022 \pm 0.001,$$

which is reasonable given the discussion in problem 2 b). For RK4 however, using a linear regression to approximate the error is not sufficient. This is because the method is a *fourth order method* (hence its name), and a linear approximation of the solution leaves an error term of $\mathcal{O}(h^2)$, by Taylor's theorem. Therefore, we use instead the 5 closest points to ξ_N, and interpolate a fourth order polynomial to the data points. For $h \gtrsim 0.005$, the log-log plot yields a linear curve from which the least squares estimate for α is

$$\alpha_{\mathrm{RK4}} \approx 3.936 \pm 0.006,$$

which is a reasonable result for the fourth order method. For very small values of h however, the error stagnates, as seen from the plots below. This is because when h is small, the rounding error in the given ξ_N dominates over the error in the RK4-integration. For comparison, if we plot the error for $n = 1$, where we know that $\xi_N = \pi$, there will be much lower rounding error in ξ_N. This shows the order of the methods much clearer, since the global error is less polluted by error in ξ_N. This is shown in figure 6. Here, we obtain

$$\alpha_{\mathrm{RK4}} \approx 4.08 \pm 0.007,$$

which is closer to what is expected from the fourth order method.

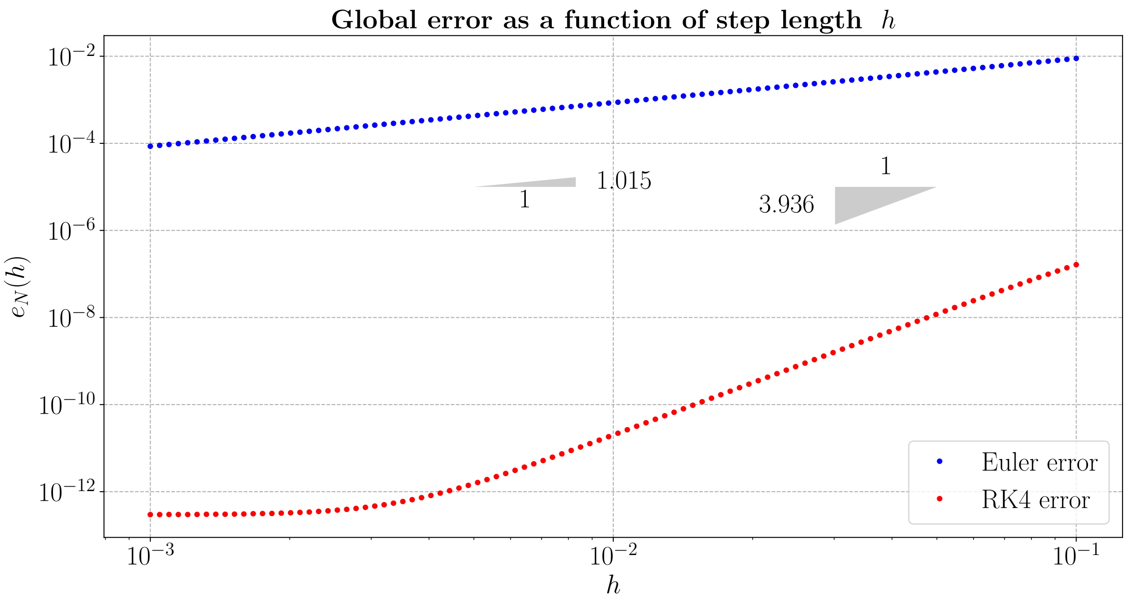

Figure 4. The figure shows the plot of the global error as a function of h for the ultra-relativistic case, $n = 3$.

Another way to avoid the problems of the rounding error in ξ_N, is to choose a $\xi^* \in (0, \xi_N)$ and specify the step lengths of interest in the interval $[h_{\min}, h_{\max}]$, and thereby calculate a reference value θ^* which is the RK4 solution evaluated at ξ^* using

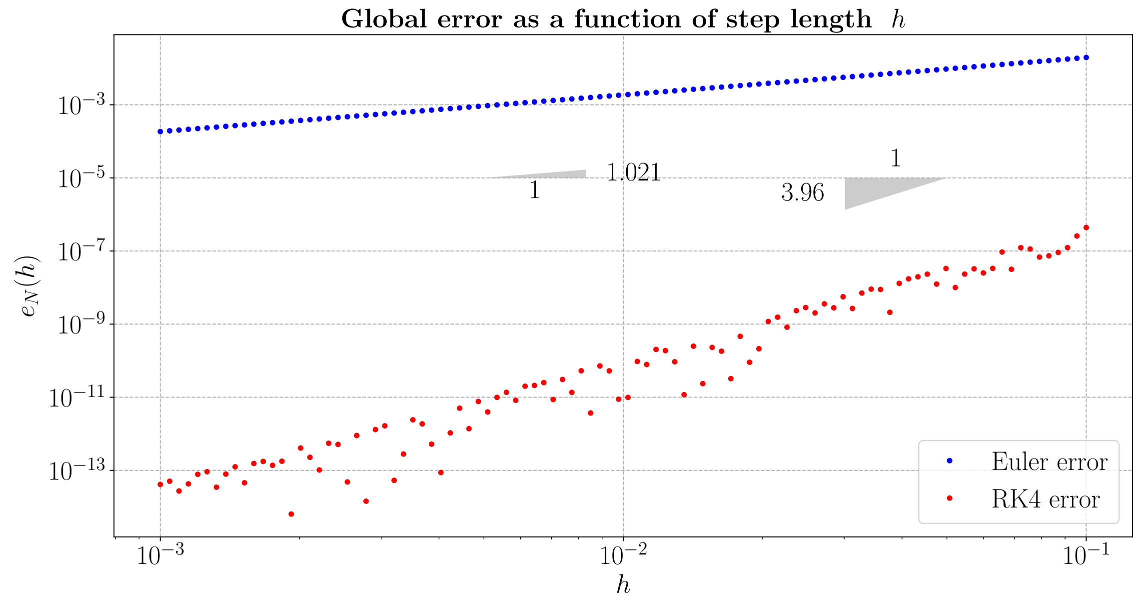

Figure 5. The figure shows the plot of the global error as a function of h for the non-relativistic case, $n = 1.5$.

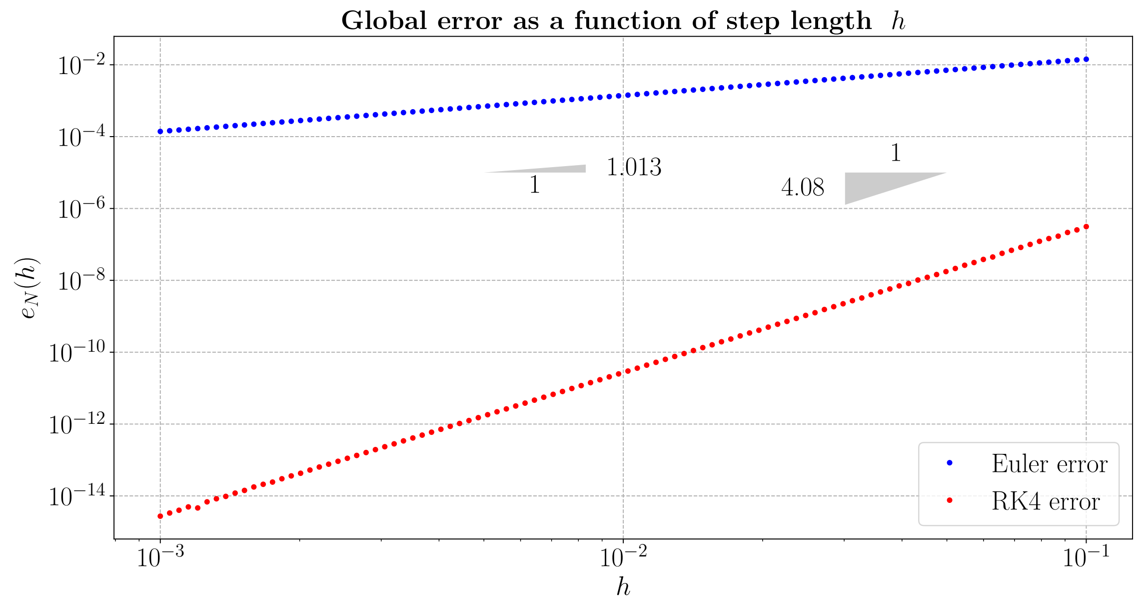

Figure 6. The figure shows the plot of the global error as a function of h for the case where $n = 1$.

for example $h_{\min}/10$ as step length. For this procedure $|\theta_N - \theta^*|$ is the measure of the error. This is of course not entirely correct, but we nonetheless avoid the stagnation of the error for small h. It should however be noted that generally would be better to specify the number of steps N and thereby calculate h so as to exactly hit ξ_N at the last point, than to do the interpolation done here. But, it does however allow for choosing h freely.

3.3 Problem 3 - Numerical solution of the TOV-equations

h) Boundary conditions

For the TOV-equation (2.3) the boundary conditions are the

$$P(R) = 0, \quad P'(0) = 0.$$

For the derivative of $\bar{P}$ this simply means that $\frac{d\bar{P}}{dx}(0) = 0$. Having an analytical solution at hand in equation (2.4), it is easily seen that

$$\bar{P}(0) = \left. \frac{\sqrt{1-\alpha x^2} - \sqrt{1-\alpha}}{3\sqrt{1-\alpha} - \sqrt{1-\alpha x^2}} \right|_{x=0} = \frac{1-\sqrt{1-\alpha}}{3\sqrt{1-\alpha}-1} = \frac{\sqrt{1-\alpha}-1}{1-3\sqrt{1-\alpha}}.$$

i) Comparison of numerical and analytical solutions of the TOV-equations

The numerical solutions below are almost inseparable from the analytical solution for $h = 0.01$. The slightly orange outline of the curves indicate that Euler's method fail to resemble the analytical solution as good as the RK4-method.

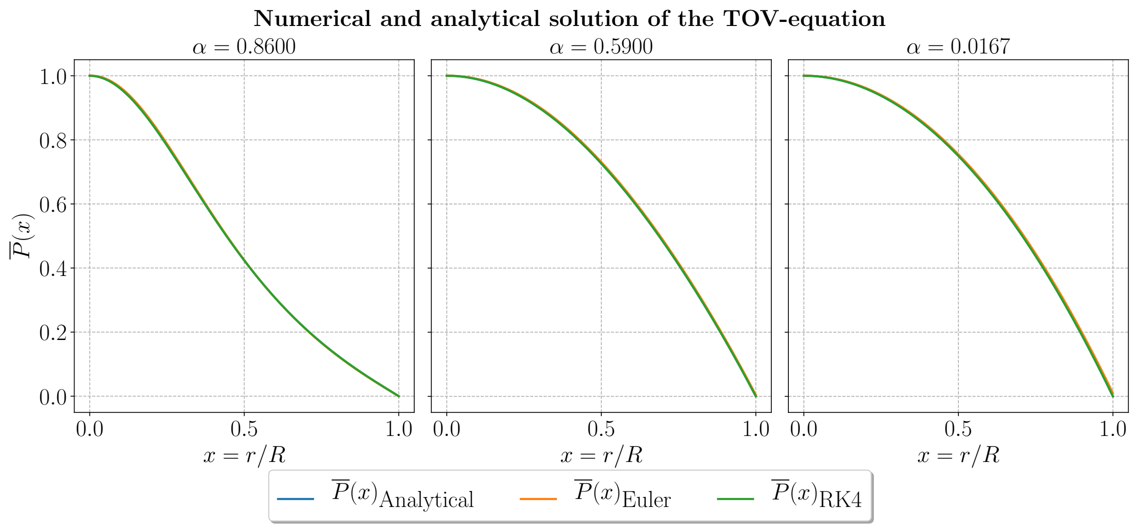

Figure 7. The figure shows the plot of the solutions of the TOV-equation with step length 0.01.

j) Discussion of the Newtonian approximation to the TOV-equation

The Newtonian approximation to the TOV-equation is what was derived in problem 2c by Taylor expanding the analytical expression when $\alpha \ll 1$. It is therefore reasonable to suspect that in this limit, the Newtonian approximation will be good. The reason for this stems from the fact that the Schwarzschild radius $r_s = \frac{2GM}{c^2}$ is the radius of a black hole of mass M[1] [2, p.31] . If the Schwarzschild radius is small compared to the actual radius of the star with mass M, this is equivalent to saying that the star must be compressed *a lot* in order to become a black hole. That is, when the ratio $\alpha \ll 1$, the star is *far from* begin a black hole, so the relativistic effects are negligible. This is indeed what is observed in figure 8. The Newtonian approximation fits not well at all for $\alpha = 0.86$, very well for $\alpha = 0.59$, and is essentially identical to the exact solution for $\alpha = 0.0167$.

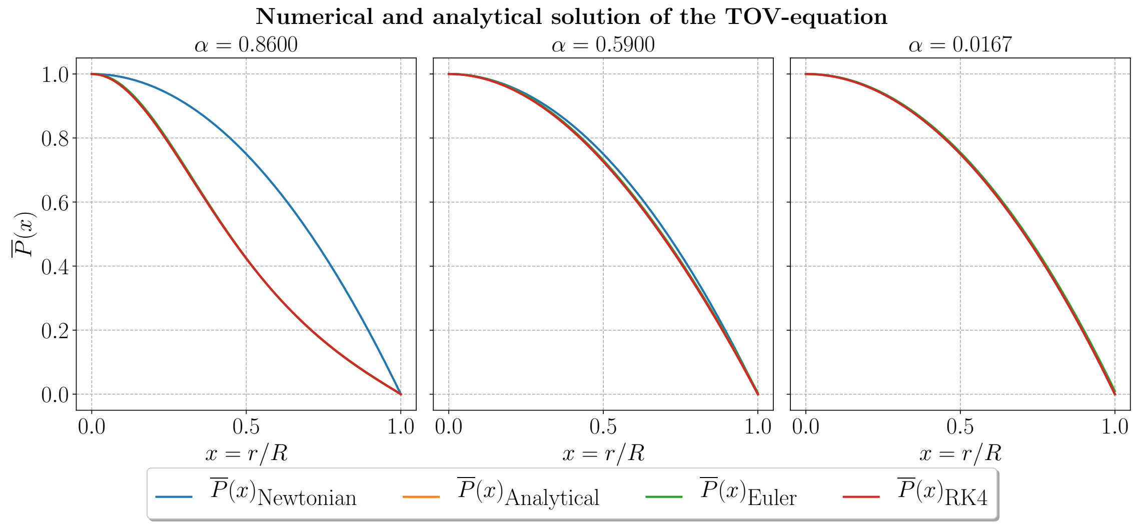

Figure 8. The figure shows the plot of the solutions of the TOV-equation with step length 0.01, with the Newtonian approximation.

References

[1] Aase, N., Andersen, J., Ballestad, T., *Kompakte stjerner* (2020)

[2] Tong, D. *Dynamics and Relativity* (2013) Cambridge, UK
http://www.damtp.cam.ac.uk/user/tong/relativity/dynrel.pdf

[1] By some lucky coincident, this happens to equal the radius of a star whose escape velocity at the surface equals c, as obtained from Newtonian mechanics.